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On the Uniform Approximation of a Class of Analytic Functions by Bruwier Series

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For a class of analytic functions $f(z)$ defined by Laplace–Stieltjes integrals the uniform convergence on compact subsets of the complex plane of the Bruwier series (B-series) $\sum_{n=0}^{\infty} \lambda_n(f) \frac{(z-nc)^n}{n!}$, $\lambda_n(f) = f^{(n)}(nc) + cf^{(n+1)}(nc)$, generated by $f(z)$ and the uniform approximation of the generating function $f(z)$ by its B-series in cones $|\arg z| \leqslant \varphi < \frac{\pi}{2}$ is shown.

1. INTRODUCTION

In order to solve the difference-differential equation $\sum_{k=0}^{m} \alpha_k f^{(k)} (z-(m-k)\, c) = 0$, $\alpha_k \in \mathbb{C}$, $c \neq 0$ L. Bruwier [1] has used series of the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n \frac{(z-nc)^n}{n!} = \sum_{n=0}^{\infty} \lambda_n b_n(z). \tag{1.1}$$

If (see [1])

$$\lambda = \limsup_{n \to \infty} |\lambda_n|^{1/n} < \frac{1}{e\,|c|} \tag{1.2}$$

this series converges uniformly on compact subsets of the complex plane $\mathbb{C}$ so that the sum $f(z)$ is an entire function. If $\lambda > \frac{1}{e\,|c|}$ the series is divergent, whereas with $\lambda = \frac{1}{e\,|c|}$ different cases can occur. In a paper about series of the form (1.1)—called Bruwier series—O. Perron [3] pointed out that Bruwier has used the uniqueness of expansion (1.1) in his studies without any proof. Therefore in [3] subsequently the uniqueness was proven and the following characterization of all entire functions $f(z)$ with a representation (1.1) was given.

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THEOREM 1 [O. Perron]. *The entire function $f(z)$ can be represented by a convergent B-series with $\lambda < \frac{1}{e\,|c|}$ if and only if the inequality*

$$|f^{(n)}(cnx)| \leqslant k \left(\frac{\alpha}{|c|} e^{-x} \right)^n, \qquad 0 < \alpha < 1, \quad k > 0 \tag{1.3}$$

is satisfied for all $x \in [0, 1]$ and $n \in \mathbb{N}_0$.

Remark 1. A completely different characterization of those entire functions $f(z)$ with a representation (1.1) will be shown in Section 2 of this paper.

Now, it is easily shown that if an entire function $f(z)$ has a representation (1.1) then the coefficients have the form

$$\lambda_n(f) = f^{(n)}(nc) + cf^{(n+1)}(nc), \qquad n \in \mathbb{N}_0 \tag{1.4}$$

e.g. all entire functions of exponential type $\sigma < \frac{1}{|c|}$ have convergent resp. Mittag–Leffler-summable representations (1.1) (see [6]).

Because of (1.4) each function—even not entire ones—can be associated with a formal B-series

$$B(z; c, f) = \sum_{n=0}^{\infty} \lambda_n(f) \frac{(z - nc)^n}{n!} \tag{1.5}$$

generated by $f(z)$ if the coefficients $\lambda_n(f)$ are defined. If a B-series converges at a point $z_0 \in \mathbb{C}$ then the series converges uniformly on compact subsets of $\mathbb{C}$ and the sum $B(z; c, f)$ is an entire function. Of course, if $f(z)$ is not entire we have $B(z; c, f) \neq f(z)$. In Section 3 (Theorem 4) it will be shown that for a class of functions analytic in the half plane $\Re z > 0$, which can be represented by Laplace–Stieltjes integrals, $B(z; c, f)$ (resp. a slightly modificated B-series, cf. (3.10)) converges for all $c > 0$ and approximates the generating function $f(z)$ uniformly on cones $|\arg z| \leqslant \varphi < \frac{\pi}{2}$ if $c > 0$ is sufficiently small.

Remark 2. In [2] W. A. J. Luxemburg discussed the approximation of functions of the form

$$f(z) = z \int_0^1 \frac{d\mu(t)}{1 + zt}, \tag{1.6}$$

$\mu(t)$ a Lebesgue–Stieltjes measure of finite total variation on [0, 1], by Abel series. For small $c > 0$ the Abel series of $f(z)$ approximates $f(z)$ uniformly in half planes $\Re z \geqslant x_0 > -1$. According to Sheffers classification of sequences of polynomials (see [5]) the interpolation polynomials of Abel

$$a_0(z) \equiv 1$$

$$a_n(z) = \frac{z(z-nc)^{n-1}}{n!}, \qquad n \in \mathbb{N}$$

and the Bruwier polynomials

$$b_n(z) = \frac{(z-nc)^n}{n!}, \qquad n \in \mathbb{N}_0$$

belong to the same class of Sheffer polynomials (see [6]). Therefore it is not surprising that similar results can be proven for B-series as well as Abel series.

2. REPRESENTATION OF ENTIRE FUNCTIONS BY B-SERIES

This problem is discussed very extensively in papers [3] and [6]. For the reader's convenience only the most important results are summarized in this section: The polynomials $b_n(z) = (z-nc)^n/n!$ are Sheffer polynomials with the generating function

$$\frac{e^{zw(t)}}{1+cw(t)} = \sum_{n=0}^{\infty} t^n b_n(z). \tag{2.1}$$

Here $w(t)$ is the absolutely smallest root of the transcendental equation $we^{cw} = t$ which can be expanded into a power series of the form

$$w(t) = \sum_{n=1}^{\infty} \frac{(-nc)^{n-1}}{n!} t^n. \tag{2.2}$$

The radius of convergence is equal to $\frac{1}{e\,|c|}$. If $\mathbf{K}_t$ is the circle $|t| < \frac{1}{e\,|c|}$ then $\mathbf{K}_t$ is mapped conformal to the compact convex domain $w(\mathbf{K}_t)$ bounded by the points w with $|cwe^{cw+1}| = 1$, $\Re cw \geqslant -1$. At $t = -\frac{1}{ec}$ $w(t)$ has a bifurcation point although the series (2.2) is convergent at this point. If $|t| < \frac{1}{e\,|c|}$ (2.1) implies

$$e^{zw} = (1+cw) \sum_{n=0}^{\infty} (we^{cw})^n\, b_n(z). \tag{2.3}$$

If $f(z)$ is an entire function of exponential type which has a conjugate indicator diagram $D(f) \subset w(\mathbf{K}_t)$ then the well known Pólya-representation of $f(z)$ implies

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\Gamma} e^{zw} F(w)\, dw \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} b_n(z) \oint_{\Gamma} (1+cw)(we^{cw})^n\, F(x)\, dw \\ &= \sum_{n=0}^{\infty} \lambda_n(f)\, b_n(z) \end{aligned} \tag{2.4}$$

with the coefficients

$$\begin{aligned} \lambda_n(f) &= f^{(n)}(nc) + cf^{(n+1)}(nc) \\ &= \frac{1}{2\pi i} \oint_{\Gamma} (1+cw)(we^{cw})^n\, F(w)\, dw \end{aligned} \tag{2.5}$$

and the Laplace–Borel transform $F(w) = \sum_{n=0}^{\infty} \frac{n!\, f_n}{w^{n+1}}$ of the entire function $f(z) = \sum_{n=0}^{\infty} f_n z^n$. Γ is a contour in $w(\mathbf{K}_t)$ containing $D(f)$ in its interior.

This result can be summarized in the following.

THEOREM 2 (see [6]). *Each entire function $f(z)$ of exponential type having a conjugate indicator diagram which is contained in the compact convex domain $w(\mathbf{K}_t)$ bounded by the points w satisfying $|cwe^{cw+1}| = 1$, $\Re cw \geqslant -1$, can be expanded into a convergent B-series.*

Remark 3. Because of Theorem 2 we now have a second characterization of all those entire functions which can be expanded into a convergent B-series, completely different from Perron's one mentioned in the introduction.

Remark 4. Conversely, the sum of convergent B-series is an entire function of exponential type whose conjugate indicator diagram lies in $w(\mathbf{K}_t)$.

3. B-SERIES OF A CLASS OF ANALYTIC FUNCTIONS

Let $\mathscr{K}$ be the class of all analytic functions $f(z)$ given by the Laplace–Stieltjes integral

$$f(z) = \int_0^{\infty} e^{-zs}\, dm(s), \tag{3.1}$$

where $m(s)$ is a complex function of bounded variation on $[0, \infty)$. That $f(z)$ determines the coefficient $\lambda_0(f) = f(0) + cf'(0)$ in addition the existence of the first moment of $m(s)$ is assumed, which means the integrals

$$M_k = \int_0^\infty s^k \, dm(s), \qquad k = 0, 1 \tag{3.2}$$

converge. Since the abscissa σ_c of convergence and σ_a of absolute convergence are not positive each function $f(z) \in \mathscr{K}$ is analytic in the half plane $\Re z > 0$ so that all coefficients $\lambda_n(f)$ are defined and a (formal) B-series—called B-series $B(z; c, f)$ generated by $f(z)$—can be associated with $f(z)$. Now, from $c > 0$ and $f(z) \in \mathscr{K}$ follows that

$$\lambda_n(f) = \int_0^\infty \lambda_n(e^{-zs}) \, dm(s) = \int_0^\infty (1 - cs)(-se^{-cs})^n \, dm(s) \tag{3.3}$$

so that we have

$$B(z; c, e^{-zs}) = (1 - cs) \sum_{n=0}^{\infty} (-se^{-cs})^n \, b_n(z).$$

Because of $0 \leqslant se^{-cs} \leqslant \frac{1}{ec}$ the B-series converges for all $s \geqslant 0$ and the sum $B(z; c, e^{-zs})$ is given by

$$B(z; c, e^{-zs}) = \begin{cases} e^{-zc}, & 0 \leqslant s < \dfrac{1}{c} \\ 0, & s = \dfrac{1}{c} \\ \dfrac{1 - cs}{1 - cv(s)} e^{-zv(s)}, & s > \dfrac{1}{c} \end{cases} \tag{3.4}$$

where $v(s)$ is the unique real solution of the equation

$$ve^{-cv} = se^{-cs}, \qquad s \geqslant \frac{1}{c}, \tag{3.5}$$

satisfying $0 \leqslant v(s) \leqslant \frac{1}{c}$ and $\lim_{s \to (1/c)+} v(s) = \frac{1}{c}$. Furthermore, because of $e^{-cv}v' = \frac{1 - cs}{1 - cv} e^{-cs} < 0$ for all $s > \frac{1}{c}$ we have

$$\lim_{s \to \infty} v(s) = 0. \tag{3.6}$$

Remark 5. (3.4) implies $\lambda_n(e^{-z/c})=0$ for all $n\in\mathbb{N}_0$ and $c\in\mathbb{C}$. But there is no non-trivial entire function $f(z)$ of exponential type $\sigma<\frac{1}{|c|}$ so that $\lambda_n(f)=0$ for all n. Therefore, the representation of entire functions of exponential type $\sigma<\frac{1}{|c|}$ by convergent resp. Mittag–Leffler summable B-series is unique so that these functions are a uniqueness class for the interpolation problem

$$\lambda_n(f)=l_n, \qquad n\in\mathbb{N}_0, \tag{3.7}$$

$\{l_n\}$ a given sequence of complex numbers (see [6] and [7]). In accordance with (3.4) $B(z;c,e^{-zs})$ has a (removable) discontinuity at $s=\frac{1}{c}$ so that the integral in

$$\begin{aligned} B(z;c,f) &= \sum_{n=0}^{\infty}\lambda_n(f)\,b_n(z)=\sum_{n=0}^{\infty}b_n(z)\int_0^{\infty}(1-cs)(-se^{-cs})^n\,dm(s)\\ &=\int_0^{\infty}\sum_{n=0}^{\infty}(1-cs)(-se^{-cs})^n\,b_n(z)\,dm(s)\\ &=\int_0^{\infty}B(z;c,e^{-zs})\,dm(s)\end{aligned}$$

which we get if we exchange summation and integration is not defined if $m(s)$ is discontinuous at $s=\frac{1}{c}$. It is easy to relieve oneself of this restriction by the following method (cf. also [4] or [7]): If we take the expression

$$f(z)=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\left(1-\frac{z}{nc}\right)^n+\alpha_\infty e^{-z/c}$$

in place of the B-series then because of

$$\left(1-\frac{z}{nc}\right)^n=\frac{n!}{(-nc)^n}\,b_n(z) \tag{3.8}$$

we get (see also Remark 7)

$$\begin{aligned}\alpha_0 &= \lambda_0(f)\\ \alpha_n &= \frac{(-nc)^n}{n!}\lambda_n(f-\alpha_\infty e^{-z/c})=\frac{(-nc)^n}{n!}\lambda_n(f), \qquad n\in\mathbb{N}\\ \alpha_\infty &= f(0)-\sum_{n=0}^{\infty}\alpha_n\end{aligned} \tag{3.9}$$

and finally

$$f(z) = \lambda_0(f)(1 - e^{-z/c}) + \sum_{n=1}^{\infty} \lambda_n(f) \frac{(-nc)^n}{n!} \left[\left(1 - \frac{z}{nc}\right)^n - e^{-z/c} \right]$$
$$+ f(0)\, e^{-z/c}$$
$$= \lambda_0(f)(1 - e^{-z/c}) + \sum_{n=1}^{\infty} \lambda_n(f) \left[b_n(z) - \frac{(-nc)^n}{n!} e^{-z/c} \right]$$
$$+ f(0)\, e^{-z/c}. \tag{3.10}$$

Now, each $f \in \mathscr{K}$ generates a series of the form (3.10) which we again call a B-series $\hat{B}(z; c, f)$.

Remark 6. If the B-series $B(z; c, f)$ converges, obviously

$$\hat{B}(z; c, f) = B(z; c, f) + [f(0) - B(0; c, f)]\, e^{-z/c} \tag{3.11}$$

is valid, that means if $B(z; c, f) = f(z)$ then follows $\hat{B}(z; c, f) = B(z; c, f) = f(z)$ so that the series $\hat{B}(z; c, f)$ can be seen as a special kind of summation of B-series.

Remark 7. Because of $|(1 - \frac{z}{nc})^n - e^{-z/c}| \leqslant |\frac{z}{c}|^2\, e^{|z/c|}/n$ the expansion (3.10) of an entire function $f(z)$ can be interpreted also as the subtraction of factors in the B-series for $f(z) - f(0)\, e^{-z/c}$ which improves the convergence.

Now, we can prove the following

THEOREM 3. *For each real $c > 0$ and $z \in \mathbb{C}$ the B-series*

$$\hat{B}(z; c, e^{-zs}) = (1 - cs) \left\{ 1 - e^{-z/c} + \sum_{n=1}^{\infty} \frac{(nc)^n}{n!} (se^{-cs})^n \right.$$
$$\left. \times \left[\left(1 - \frac{z}{nc}\right)^n - e^{-z/c} \right] \right\} + e^{-z/c} \tag{3.12}$$

generated by e^{-zs} converges uniformly and absolutely with respect to real $s \geqslant 0$. Furthermore, we have

$$\hat{B}(z; c, e^{-zs}) = \begin{cases} e^{zs}, & 0 \leqslant s \leqslant \dfrac{1}{c} \\ \dfrac{1 - cs}{1 - cv(s)} [e^{-zv(s)} - e^{-z/c}] + e^{-z/c}, & s > \dfrac{1}{c} \end{cases} \tag{3.13}$$

where $v(s)$ *is the unique real solution of the equation* $ve^{-cv}=se^{-cs}$, $s\geqslant\frac{1}{c}$, *satisfying* $v(s)\leqslant\frac{1}{c}$.

Proof. It is easy to check the uniform and absolute convergence of B-series (3.12), so that we only have to show formula (3.13). For $0\leqslant s<\frac{1}{c}$ the B-series

$$B(z;c,e^{-zs})=(1-cs)\sum_{n=0}^{\infty}(-se^{-cs})^n\,b_n(z) \tag{3.14}$$

represents (cf. (3.4)) the function e^{-zs}. If $z=0$ we have $B(0;c,e^{-zs})=1$ which together with (3.11), immediately implies $\hat{B}(z;c,f)=e^{-zs}$ if $0\leqslant s\leqslant\frac{1}{c}$. For $s>\frac{1}{c}$ the series (3.14) represents in accordance with (3.4) the function $\frac{1-cs}{1-cv(s)}e^{-zv(s)}$ so that again with (3.11)

$$\hat{B}(z;c,e^{-zs})=\frac{1-cs}{1-cv(s)}\,[e^{-zv(s)}-e^{-z/c}]+e^{-z/c}$$

follows. ∎

Remark 8. Because of the uniform convergence in $s\geqslant0$ $\hat{B}(z;c,e^{-zs})$ is continuous also at $s=\frac{1}{c}$, which can also be seen directly from (3.13).

The main result of the paper is the following.

THEOREM 4. *If* $f(z)\in\mathscr{K}$ *the B-series*

$$\hat{B}(z;c,f)=\sum_{n=0}^{\infty}\lambda_n(f)\,\frac{(-nc)^n}{n!}\left[\left(1-\frac{z}{nc}\right)^n-e^{-z/c}\right]+f(0)\,e^{z/c},\qquad 0^0:=1 \tag{3.15}$$

generated by $f(z)$ *converges for each* $c>0$ *absolutely and uniformly on compact subsets of the complex plane. In the half plane* $\Re z>0$

$$\hat{B}(z;c,f)=f(z)+e^{-z/c}\int_{1/c}^{\infty}\left\{\frac{1-cs}{1-cv(s)}\,[e^{(z/c)(1-cv(s))}-1]\right.$$
$$\left.+1-e^{(z/c)(1-cs)}\right\}\,dm(s), \tag{3.16}$$

and the estimate

$$|\hat{B}(z; c, f) - f(z)| \leqslant \left|\mu(\infty) - \mu\left(\frac{1}{c}\right)\right| \left[d_1 + \left(\frac{|\Im z|}{c} + d_2 \frac{|\Im z|^2}{c^2}\right) e^{-\Re z/c}\right] + 2e^{-\Re z/c} \int_{1/c}^{\infty} |dm(s)| \tag{3.17}$$

$$d_1, d_2 > 0, \qquad \mu(s) = \int_0^s (1 - cu)\, dm(u), \qquad \mu(\infty) = M_0 - cM_1,$$

are valid. Furthermore, if $c \to 0+$ $\hat{B}(z; c, f)$ *tends to* $f(z)$ *uniformly in each cone* $|\arg z| \leqslant \varphi < \frac{\pi}{2}$.

Proof. If $f(z) \in \mathscr{K}$ we have

$$\lambda_n(f) = (-1)^n \int_0^{\infty} (1 - cs)(se^{-cs})^n\, dm(s)$$

which together with

$$s^{n+1} e^{-cns} \leqslant \frac{2}{c} (ec)^{-n}$$

immediately implies

$$|\lambda_n(f)| \leqslant \frac{3}{(ec)^n} \int_0^{\infty} |dm(s)|.$$

From this inequality and $|(1 - \frac{z}{nc})^n - e^{-z/c}| \leqslant |\frac{z}{c}|^2\, e^{|z/c|}/n$ follows that the B-series $\hat{B}(z; c, f)$ generated by $f(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}$. Therefore the sum $\hat{B}(z; c, f)$ is an entire function of the form (cf. Theorem 3, (3.12))

$$\begin{aligned}\hat{B}(z; c, f) &= \int_0^{\infty} \left\{(1 - cs) \sum_{n=0}^{\infty} (se^{-cs})^n \frac{(nc)^n}{n!} \left[\left(1 - \frac{z}{nc}\right)^n - e^{-z/c}\right]\right\} dm(s) \\ &\quad + f(0)\, e^{-z/c} \\ &= \int_0^{\infty} [\hat{B}(z; c, e^{-zs}) - e^{-z/c}]\, dm(s) + f(0)\, e^{-z/c} \\ &= \int_0^{\infty} \hat{B}(z; c, e^{-zs})\, dm(s). \end{aligned} \tag{3.18}$$

The last integral in (3.18) exists since (cf. (3.13))

$$\hat{B}(z;\, c,\, e^{-zs}) = O(s) \qquad \text{if} \quad s \to \infty.$$

Now, (3.13) implies

$$\hat{B}(z;\, c,\, f) = \int_0^{1/c} e^{-zs}\, dm(s) + \int_{1/c}^{\infty} \left\{ \frac{1-cs}{1-cv(s)}\, [e^{-zv(s)} - e^{-z/c}] + e^{-z/c} \right\} dm(s)$$

which gives (3.16) respectively

$$\hat{B}(z;\, c,\, f) - f(z) = \int_{1/c}^{\infty} \frac{1-cs}{1-cv(s)}\, [1 - e^{-(s/c)(1-cv(s))}]\, e^{-zv(s)}\, dm(s)$$

$$+ \int_{1/c}^{\infty} (e^{-z/c} - e^{-zs})\, dm(s)$$

$$= I_1(z, c) + I_2(z, c)$$

and

$$|\hat{B}(z;\, c,\, f) - f(z)| \leqslant |I_1(z, c)| + |I_2(z, c)|. \tag{3.19}$$

It is easy to check that

$$|I_2(z, c)| \leqslant 2e^{-\Re z/c} \int_{1/c}^{\infty} |dm(s)| \tag{3.20}$$

so that we only have to estimate $I_1(z, c)$. If we rewrite $I_1(z, c)$ to

$$I_1(z, c) = \int_{1/c}^{\infty} (1-cs)\, e^{-zv(s)} \left(\int_0^{z/c} e^{-t(1-cv(s))}\, dt \right) dm(s)$$

$$= \int_0^{z/c} e^{-t} \left(\int_{1/c}^{\infty} (1-cs)\, e^{-v(s)[z-ct]}\, dm(s) \right) dt$$

we get with $z = x + iy$ (the integrand is entire with respect to t)

$$I_1(z, c) = \int_0^{x/c} e^{-t} \left(\int_{1/c}^{\infty} (1-cs)\, e^{-v(s)[x-ct]}\, dm(s) \right) dt$$

$$+ ie^{-x/c} \int_0^{y/c} e^{-it} \left(\int_{1/c}^{\infty} (1-cs)\, e^{-iv(s)[y-ct]}\, dm(s) \right) dt$$

$$= I_1(x, c) + e^{-x/c} I_1(iy, c). \tag{3.21}$$

With $\mu(s)=\int_0^s (1-cu)\,dm(u)$ (note that $\mu(s)$ is bounded for $s\geqslant\frac{1}{c}$ and $|v'(s)|=-v'(s)$) the estimate

$$\left|\int_{1/c}^{\infty}(1-cs)\,e^{-v(s)[x-ct]}\,dm(s)\right|$$

$$=\left|\int_{1/c}^{\infty}e^{-v(s)[x-ct]}\,d\left(\mu(s)-\mu\left(\frac{1}{c}\right)\right)\right|$$

$$\leqslant\left|\mu(\infty)+\mu\left(\frac{1}{c}\right)\right|$$

$$+(x-ct)\left|\int_{1/c}^{\infty}v'(s)\,e^{-v(s)[x-ct]}\left(\mu(s)-\mu\left(\frac{1}{c}\right)\right)ds\right|$$

$$\leqslant\left|\mu(\infty)+\mu\left(\frac{1}{c}\right)\right|+\left|\mu(\xi)-\mu\left(\frac{1}{c}\right)\right|$$

$$\leqslant d_1\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|,\ \xi\in\left[\frac{1}{c},\infty\right]$$

follows, so that we finally get

$$|I_1(x,c)|\leqslant d_1\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|\int_0^{x/c}e^{-t}\,dt$$

$$\leqslant d_1\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|. \tag{3.22}$$

The estimate

$$\left|\int_{1/c}^{\infty}(1-cs)\,e^{-iv(s)[y-ct]}\,dm(s)\right|$$

$$\leqslant\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|+|y-ct|\int_{1/c}^{\infty}|v'(s)|\left|\mu(s)-\mu\left(\frac{1}{c}\right)\right|ds$$

$$\leqslant\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|+\frac{M}{c}|y-ct|\leqslant\left|\mu(\infty)-\mu\left(\frac{1}{c}\right)\right|\left(1+d_2\frac{|y|}{c}\right),$$

$$M=\max_{s\in[1/c,\,\infty]}\left|\mu(s)-\mu\left(\frac{1}{c}\right)\right|$$

implies

$$|I_1(iy, c)| \leqslant \left|\mu(\infty) - \mu\left(\frac{1}{c}\right)\right| \left(\frac{|y|}{c} + d_2 \frac{|y|^2}{c^2}\right) \tag{3.23}$$

so that

$$|I_1(z, c)| \leqslant \left|\mu(\infty) - \mu\left(\frac{1}{c}\right)\right| \left[d_1 + \left(\frac{|y|}{c} + d_2 \frac{|y|^2}{c^2}\right) e^{-x/c}\right] \tag{3.24}$$

is valid. Now, (3.20) and (3.24) imply inequality (3.17) of the theorem. To finish the proof let us choose $c > 0$ so that $|\mu(s) - \mu(s^*)| < \varepsilon$ for arbitrary $\varepsilon > 0$ and all s, $s^* \geqslant \frac{1}{c}$. This is always possible since the existence of the integrals $\int_0^\infty s^k\, dm(s)$, $k = 0, 1$, is assumed (cf. (3.2)). Now, estimate (3.17) immediately shows the rest of the theorem. ∎

Remark 9. It is easily checked that the class of functions (1.6) examined by W. A. J. Luxemburg in connection with Abel series (cf. [2] resp. Remark 2) generates convergent B-series $B(z; c, f)$ and $\hat{B}(z; c, f)$. With help of the representation

$$f(z) = \int_0^1 \left(\int_0^\infty \frac{1 - e^{-zst}}{t} e^{-s}\, ds\right) d\mu(t)$$

it can be shown that (for sufficient small $c > 0$) $f(z)$ is approximated uniformly by $B(z; c, f)$ or $\hat{B}(z; c, f)$ on half planes $\Re z \geqslant x_0 > -1$ also.

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